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Note

A Note on Strong Uniqueness Constants

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We expand a result of Blatt, concerning the strong uniqueness constants of uniform best approximations on $[-1, 1]$.

Let X be compact, $f \in C(X)$, and V be a subspace of $C(X)$, the space of all real-valued continuous functions on X. Let $v \in V$ be a best uniform approximation to $f \in C(X)$, i.e., a best approximation with respect to the uniform norm

$$\|r\| = \|r\|_X := \max_{x \in X} |r(x)|.$$

v is then called a strongly unique best approximation to f, if there is a $\gamma > 0$ with

$$\|f - w\| \geqslant \|f - v\| + \gamma \|v - w\| \qquad \text{for all} \quad w \in V. \tag{1}$$

The largest constant satisfying (1) is called the strong uniqueness constant $\gamma(f)$ and (see [2])

$$\gamma(f) = \inf_{w \in V,\ \|w\| = 1} \ \sup_{x \in E(r)} (\operatorname{sgn} r(x))\, w(x), \tag{2}$$

where $r = f - v$ and

$$E(r) := \{x \in X\colon |r(x)| = \|r\|\}.$$

Following [5] we define for a Banach space E and a subspace $U \subseteq E$

$$\lambda(U, E) := \inf\{\|p\|\colon p\colon E \to U \text{ is a projection}\}.$$

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THEOREM 1. *Let v be a strongly unique best approximation to f. Then*

$$\gamma(f) \leqslant \frac{\lambda(V|_{E(r)}, C(E(r)))}{\lambda(V, C(X))}.$$

Proof. By (2)

$$\gamma(f) \leqslant \inf_{w \in V, \|w\| = 1} \|w\|_{E(r)}$$

$$= (\sup\{\|w\| : \|w\|_{E(r)} \leqslant 1, w \in V\})^{-1}.$$

Let $T: C(E(r)) \to V|_{E(r)}$ be any projection. Define $\phi: V|_{E(r)} \to V$ by

$$\phi(w|_{E(r)}) = w.$$

ϕ is well defined, since otherwise there is a $w \in V$ with $\|w\| = 1$ and $w|_{E(r)} = 0$, contradicting $\gamma(f) > 0$. Define for $f \in C(X)$

$$S(f) := \phi(T(f|_{E(r)})).$$

Then

$$\lambda(V, C(X)) \leqslant \|S\| \leqslant \|\phi\| \; \|T\| \leqslant \gamma(f)^{-1} \; \|T\|.$$

Thus

$$\lambda(V, C(X)) \leqslant \gamma(f)^{-1} \, \lambda(V|_{E(r)}, C(E(r))).$$

This proves Theorem 1. ∎

COROLLARY 1. *Let $V = \Pi_n$, $X = [a, b]$, and $|E(r)| = m + n + 1$. Then*

$$\gamma(f) = \gamma_n(f) \leqslant \frac{1 + \sqrt{m}}{\log(n+1)} \cdot \frac{\pi^2}{4}.$$

Proof. Since $\dim \Pi_n|_{E(r)} = n + 1$ and $\dim C(E(r)) = n + m + 1$, we get from [5, p. 341]

$$\lambda(\Pi_n|_{E(r)}, C(E(r))) \leqslant \left(\frac{n+1}{n+m+1} + \frac{\sqrt{(n+m)(n+1)}}{n+m+1} \cdot \sqrt{m}\right)$$

$$\leqslant 1 + \sqrt{m}.$$

It is a well-known result (see, e.g., [6]) that

$$\lambda(V, C(X)) \geqslant \frac{4}{\pi^2} \log(n+1).$$

Thus the result follows from Theorem 1. ∎

This is sharper than a result of Blatt [1]. Furthermore, the proof is easier, since we used the functional analytic result of König *et al.* in Theorem 1.

COROLLARY 2. *Let* $f \in C^{\infty}[a, b]$, *and assume that* $f^{(n+1)}$ *has* $o(\log(n)^2)$ *zeros in* $[a, b]$ *(as* $n \to \infty$*). Then*

$$\liminf_{n \to \infty} \gamma_n(f) = 0. \tag{3}$$

Proof. With the equioscillation theorem [6] and multiple application of Rolle's theorem, one can see that

$$|E(r_n)| = n + o(\log(n)^2),$$

where r_n denotes the error function of the best approximation with respect to Π_n. The result follows now from Corollary 1. ∎

Corollary 2 gives a partial answer to a question which was raised by Poreda [7]. It has been conjectured in [4], that all nonpolynomial $f \in C(X)$ satisfy (3). A further result in this direction can be found in [3].

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